Coq Summer School, Session 9 : Dependent programs with logical parts

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- A few types with restrictions:

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Inductive bnat (n : nat) : Type :=
  cb : forall m, m < n -> bnat n.

Inductive array (n : nat) : Type :=
  ca : forall l : list Z, length l = n -> array n.
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- We'll see now more constructions for programs with rich specifications (i.e. types)

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- For instance:

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Definition bnat n := { m | m < n }.
Definition array n := { l : list Z | length l = n }.
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A generic notion of type with restriction

- Behind the nice `{ | }` notation, the `sig` type:

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- To build a `sig` interactively: the `exists` tactic.

A example: bounded successor

- As a function:

```
Definition bsucc n : bnat n -> bnat (S n) :=
 fun m => let (x,p):= m in exist _ (S x) (lt_n_S _ _ p).
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 auto with arith.
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- Via the Program framework :

```
Program Definition bsucc n : bnat n -> bnat (S n) :=
 fun m => S m.
Next Obligation.
 destruct m. simpl. auto with arith.
Qed
```

General shape of a rich specification

- With `sig`, we can hence express also *post-conditions*:

```
forall x, P x -> { y | Q x y }
```

- Adapt to your needs: multiple arguments or outputs (`y` can be a tuple) or pre or post (`Q` can be a conjonction).
- Apart with Program, `sig` is rarely used for pre-conditions.

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- More convenient: `sumbool`, a type with two alternatives and annotations for characterizing them.

```
Definition eq_nat_dec :
 forall n m : nat, { n=m }+{ n<>m }.
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- Behind the `{ }+{ }` notation, the `sumbool` type:

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Inductive sumbool (A B : Prop) : Type :=
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Decidability result

- Many Coq functions are currently formulated this way: `eq_nat_dec`, `Z_eq_dec`, `le_lt_dec`, ...
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 induction n.
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- For equality, see tactic decide equality.

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- Pure & efficient Ocaml/Haskell code can be obtained by extraction.
- Definitions by tactics are dreadful, Program helps but is still quite experimental.
- Instead of destructing rich objects, other technics can also be convenient (iff, reflect).

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In fact, `sig`/`sumbool` live in a different world than `ex`/`or`.

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  - a program : a type : `Type`
  - `O` : `nat` : `Type`
  - `pred` : `nat->nat` : `Type`

The two worlds of Coq

Usually we program in Type and make proofs in Prop. But that's just a convention. We can build functions by tactics, or reciprocally "program" a proof:

```
Definition or_sym A B : A\/B -> B\/A :=
 fun h => match h with
  | or_introl a => or_intror _ a
  | or_intror b => or_introl _ b
 end.
```

The similarity between proofs and programs, between statements and types is called the Curry-Howard isomorphism.

The two worlds of Coq

In Coq, a rigid separation between Prop and Type:

Logical parts should not interfere with computations in Type.

```
Definition nat_of_or A B : A\/B -> nat :=
 fun h => match h with
  | or_introl _ => 0
  | or_intror _ => 1
 end.
Error: ... proofs can be eliminated only to build proofs.
```

Idea: proofs are there only as guarantee, we're interested only in their *existence*, we consider them as having no *computational content*.

Extraction

Coq's strict separation between Prop and Type is the fondation of the *extraction* mechanism: roughly, logical parts are removed, pruned programs still compute the same outputs.

```
Coq < Recursive Extraction le_lt_dec.

type nat = O | S of nat
type sumbool = Left | Right

(** val le_lt_dec : nat -> nat -> sumbool **)

let rec le_lt_dec n m =
  match n with
    | O -> Left
    | S n0 -> (match m with
                 | O -> Right
                 | S m0 -> le_lt_dec n0 m0)
```