Resource Calculi
Some Syntax, Some Semantics

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Outline

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Why Resource Calculi?

Resource Calculi

Programming languages giving a major control on the resources needed by a program during his execution.

Resources to be bounded can be of very different kinds:

- *time/space:* important for programs running in environments with bounded resources.
- *non-replicable data:* naturally arising in quantum computing (just an analogy).

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Some non-optional ingredients

We introduce in the λ-calculus *depletable arguments*:

- depletable resources are present in a limited number,
- depletable resources must be consumed.

Depletable Arguments $\Rightarrow$ Linear Substitution:

$$M\langle N/x\rangle = \text{ exactly one occurrence of } x \text{ in } M \text{ is substituted by } N$$

Depletable Arguments $\Rightarrow$ Non-Determinism:

$$(\lambda x.xx)N^{\ell} = \text{ which occurrence of } x \text{ will be substituted?}$$

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$$\begin{array}{ccccc}
(\lambda x.\langle x, x\rangle)M & \longleftarrow & (\lambda \mathbf{x}.\langle \mathbf{x}, \mathbf{x}\rangle)(\mathbf{M} + \mathbf{N}) & \longrightarrow & (\lambda x.\langle x, x\rangle)N \\
\downarrow & & \downarrow h & & \downarrow \\
\langle M, M\rangle & \overset{h}{\longleftarrow} & \langle (M + N), (M + N)\rangle & \overset{h}{\longrightarrow} & \langle N, N\rangle \\
 & \swarrow h & & h \searrow & \\
\langle M, N\rangle & & & & \langle N, M\rangle
\end{array}$$

Previously, on Resource calculus - 1993

Lambda calculus with multiplicities

Gérard Boudol. The lambda-calculus with multiplicities. INRIA Research Report 2025, 1993.

- Arguments may come in limited availability, and mixed together. They are grouped in ‘bags’.
- Lazy operational semantics + Explicit substitution.
- **Main motivation**: finer observational equivalence on classic λ-calculus.
- Boudol left for future work links with Girard’s LL…

Previously, on Resource calculus - 2003

The *differential* λ-calculus.

T. Ehrhard and L. Regnier. The differential λ-calculus. Theoretical Computer Science 2003.

- Calculus with syntactic differential operators (*linear approximations*).
- Non-lazy reduction.
- Non-determinism as formal sums of terms ($\Sigma_i M_i \not\to M_i$).
- Issued from semantic investigations (finiteness spaces).
- Original syntax quite heavy (now a little better. . .).

Previously, on Resource calculus - 2006

Taylor Expansion

T. Ehrhard and L. Regnier. Böhm trees, Krivine's machine and the Taylor expansion of λ-terms. In CiE, LNCS, 2006.

- The target of Taylor Expansion of ordinary λ-terms.

$$(MN)^* = \sum_{n=0}^{\infty} \frac{1}{n!} M^*[N^*]^n$$

The Resource Calculus - Today

Full (non-lazy/non-linear) resource calculus.

P. Tranquilli. Intuitionistic differential nets and lambda calculus. To appear in Theoretical Computer Science.

- Convincing link with differential linear logic.

Morally mixing ‘differential’ and ‘with multiplicities’ λ-calculus .

Differential λ-calculus	Resource λ-calculus
differentiation	linear substitution
two kinds of applications	two kinds of resources
heavy syntax	better syntax

Until now, no abstract model theory!

Outline

The Syntax

terms:	$M ::=$	$x \mid \lambda x.M \mid MP$	(as in λ-calculus)
bags:	$P ::=$	$[M_1, \ldots, M_m, N_1^!, \ldots, N_n^!]$	(multisets)
sums:	$\pi ::=$	$M_1 + \cdots + M_m$	(0 neutral element)

Idea:

- $[N]$ is a 'linear' argument (available *exactly* once),
- $[N^!]$ is a classic argument (available how many times you wish).

Ordinary λ-calculus: $MN \equiv M[N^!]$.

Reduction Rules (Informally)

Informal definition of reduction:

$(\lambda x.M)[N] \to$ M where N substitutes **exactly one** occurrence of x

Examples:

- Nice terms: $(\lambda x.x)[L] \to L$,
- Starving terms: $(\lambda x.yx)[] \to 0$,
- Greedy terms: $(\lambda x.y)[L] \to 0$ (we can't erase linear resources).

Two kinds of 'unsolvable':
$\Omega = (\lambda x.x[x^!])[(\lambda x.x[x^!])^!] = \text{non-termination}, \qquad 0 = \text{clash}$

- Non determinism: $(\lambda x.M[N])[L] =$ two possibilities!

Will we have sums everywhere?

Hopefully not! Sums are pushed to surface:

$$\begin{array}{rcll}
\lambda x.(M+N) & = & \lambda x.M + \lambda x.N & \\
(M+N)P & = & MP + NP & \text{(function position is } \textit{linear}\text{)} \\
M([N+L] \uplus P) & = & M([N] \uplus P) + M([L] \uplus P) & \\
M([(N+L)^!] \uplus P) & = & M([N^!, L^!] \uplus P) &
\end{array}$$

...and their 0-ary versions:

$$\begin{array}{rcl}
\lambda x.0 = 0 & \qquad & M([0] \uplus P) = 0 \\
0P = 0 & \qquad & M([0^!] \uplus P) = MP
\end{array}$$

Two kinds of substitutions

- $M\{N/x\}$: usual capture free substitution.
- $M\langle N/x\rangle$: *linear* substitution ($\cong$ *differential* operator)

On terms:

$$y\langle N/x\rangle = \begin{cases} N & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

$$(\lambda y.M)\langle N/x\rangle = \lambda y.M\langle N/x\rangle$$
$$(MP)\langle N/x\rangle = M\langle N/x\rangle P \ + \ M(P\langle N/x\rangle)$$

On Bags:

$$[]\langle N/x\rangle = 0$$
$$[M]\langle N/x\rangle = [M\langle N/x\rangle]$$
$$[M^!]\langle N/x\rangle = [M\langle N/x\rangle, M^!]$$
$$(P \uplus R)\langle N/x\rangle = P\langle N/x\rangle \uplus R \ + \ P \uplus (R\langle N/x\rangle)$$

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Reduction Rules (formally)

Giant step:

$$(\lambda x.M)[N_1,\ldots,N_n,M_1^!,\ldots,M_m^!] \to_g M\langle N_1/x\rangle\cdots\langle N_n/x\rangle\{\Sigma_i M_i/x\}$$

Theorem [Pagani-Tranquilli APLAS'09]

- $\to_g$ is confluent.
- $\to_g$ enjoys a standardization property.

Simple Type System

$$(\mathrm{Rx})\ \frac{\Gamma(x) = \sigma}{\Gamma \vdash_R x : \sigma} \qquad (\mathrm{R}\lambda)\ \frac{\Gamma, x : \sigma \vdash_R M : \tau}{\Gamma \vdash_R \lambda x.M : \sigma \to \tau}$$

$$(\mathrm{R@})\ \frac{\Gamma \vdash_R M : \sigma \to \tau \quad \Gamma \vdash_R P : \sigma}{\Gamma \vdash_R MP : \tau}$$

$$(\mathrm{Rb})\ \frac{\Gamma \vdash_R N : \sigma \quad \Gamma \vdash_R P : \sigma}{\Gamma \vdash_R [N^{(!)}] \uplus P : \sigma} \qquad (\mathrm{R}[])\ \frac{}{\Gamma \vdash_R [] : \sigma}$$

$$(\mathrm{R+})\ \frac{\Gamma \vdash_R A_i : \sigma \quad \text{for all } i}{\Gamma \vdash_R \Sigma_i A_i : \sigma}$$

Outline

The differential λ-calculus: Syntax

Differential Lambda Terms:

$$s, t ::= x \mid \lambda x.s \mid st \mid D(s) \cdot t \mid s + t \mid 0$$

Reduction Rules ($\to_D = \to_\beta \cup \to_{\beta_D}$):

$$(\beta) \qquad (\lambda x.s)t \to_\beta s\{t/x\}$$
$$(\beta_D) \qquad D(\lambda x.s) \cdot t \to_{\beta_D} \lambda x.\frac{\partial s}{\partial x} \cdot t$$

Ideas:

- st = usual application of λ-calculus $(\cong s[t^!])$
- $D(\cdots(D(s) \cdot t_1) \cdots) \cdot t_k$ = linear application $(\cong s[t_1, \ldots, t_k])$
- $\frac{\partial s}{\partial x} \cdot t$ = differential substitution $(\cong s\langle t/x \rangle)$
    - $\frac{\partial (su)}{\partial x} \cdot t = (\frac{\partial s}{\partial x} \cdot t)u + (D(s) \cdot (\frac{\partial u}{\partial x} \cdot t))u$

      $(\cong (s[u^!])\langle t/x \rangle = s\langle t/x \rangle[u^!] + s[u\langle t/x \rangle, u^!])$

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Translation between the two calculi

We can define a translation map

$$(\cdot)^o : \text{Resource calculus} \to \text{Differential } \lambda\text{-calculus}$$

- $x^o = x$,
- $(\lambda x.M)^o = \lambda x.M^o$,
- $((\lambda x.M)[\vec{L}, \vec{N}^!])^o = (D^k(\lambda x.M^o) \cdot L_1^o \cdots L_k^o)(\Sigma_i N_i^o)$,
- $0^o = 0$,
- $(\Sigma_i M_i)^o = \Sigma_i M_i^o$.

The translation is 'faithful'

For M, N resource terms: $M \to_g N$ implies $M^o \to_D^\star N^o$

Simple Types in Differential Calculus

$$x\ \frac{\Gamma(x) = \sigma}{\Gamma \vdash_D x : \sigma} \qquad \lambda\ \frac{\Gamma; x : \sigma \vdash_D s : \tau}{\Gamma \vdash_D \lambda x.s : \sigma \to \tau}$$

$$@\ \frac{\Gamma \vdash_D s : \sigma \to \tau \quad \Gamma \vdash_D t : \sigma}{\Gamma \vdash_D st : \tau} \qquad D\ \frac{\Gamma \vdash_D s : \sigma \to \tau \quad \Gamma \vdash_D t : \sigma}{\Gamma \vdash_D D(s) \cdot t : \sigma \to \tau}$$

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Remark: Linear application does not decrease types.

The translation remains 'faithful'

Let M be a resource term. If $\Gamma \vdash_R M : \sigma$ then $\Gamma \vdash_D M^o : \sigma$

What about semantics?

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(Canadian) Differential Categories

The differential λ-calculus inspired researchers working on category theory.

- Aim: Axiomatize a differential operator $D(-)$ categorically.

Differential categories

Blute, Cockett and Seely proposed:

- BCS'06: (monoidal) differential categories
  (= point of view too fine)
- BCS'09: Cartesian differential categories
  (= lack of higher order functions)

Not enough for modeling the differential λ-calculus!!!

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Left Additive Categories

We need sums on morphisms.

A category **C** is *left-additive* if:

- each homset has a structure of commutative monoid $(\mathbf{C}(A,B), +_{AB}, 0_{AB})$,
- $(g+h)\circ f = (g\circ f) + (h\circ f)$ and $0\circ f = 0$.

When f satisfies also $f\circ(g+h) = (f\circ g) + (f\circ h)$ and $f\circ 0 = 0$ it is called *additive*. (weak form of linearity)

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Cartesian (Closed) Left-additive Categories

A category **C** is *Cartesian left-additive* if:

- **C** is a left-additive category,
- it is Cartesian (=it has products),
- all projections and pairings of additive maps are additive.

A category **C** is *Cartesian closed left-additive* if:

- **C** is Cartesian left-additive,
- it is a ccc ($\Lambda(-) =$ curry, ev =eval),
- it satisfies $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$ and $\Lambda(0) = 0$.
  (implies $ev \circ \langle f+g, h\rangle = ev \circ \langle f, h\rangle + ev \circ \langle g, h\rangle$)

Cartesian Differential Categories

$$D\frac{f : A \to B}{D(f) : \underline{A} \times A \to B}$$

Satisfying:

D1. $D(f+g) = D(f) + D(g)$ and $D(0) = 0$

D2. $D(f) \circ \langle h+k, v\rangle = D(f) \circ \langle h, v\rangle + D(f) \circ \langle k, v\rangle$ and $D(f) \circ \langle 0, v\rangle = 0$

D3. $D(Id) = \pi_1$, $D(\pi_1) = \pi_1 \circ \pi_1$ and $D(\pi_2) = \pi_2 \circ \pi_1$

D4. $D(\langle f, g\rangle) = \langle D(f), D(g)\rangle$

D5. $D(f \circ g) = D(f) \circ \langle D(g), g \circ \pi_2\rangle$

D6. $D(D(f)) \circ \langle\langle g, 0\rangle, \langle h, k\rangle\rangle = D(f) \circ \langle g, k\rangle$

D7. $D(D(f)) \circ \langle\langle 0, h\rangle, \langle g, k\rangle\rangle = D(D(f)) \circ \langle\langle 0, g\rangle, \langle h, k\rangle\rangle$

Differential λ-Categories (ccc's)

[BEM'10] **C** is a *Differential λ-category* if:

- **C** is a Cartesian differential category,
- it is Cartesian closed left-additive,
- it satisfies the following rules:

For all $f : C \times A \to B$:

$$D(\Lambda(f)) = \Lambda(D(f) \circ \langle \pi_1 \times 0_A, \pi_2 \times Id_A \rangle)$$

For all $h : C \to [A \Rightarrow B]$ and $g : C \to A$:

$$D(ev \circ \langle h, g \rangle) = ev \circ \langle D(h), g \circ \pi_2 \rangle + D(\Lambda^-(h)) \circ \langle \langle 0_C, D(g) \rangle, \langle \pi_2, g \circ \pi_2 \rangle \rangle$$

Categorical Interpretation

Define $f \star g = D(f) \circ \langle\langle 0, g \circ \pi_1 \rangle, Id \rangle$:

$$\star \frac{f : C \times A \to B \quad g : C \to A}{f \star g : C \times A \to B}$$

Define $[\![\Gamma \vdash_D s : \sigma]\!] = [\![s^\sigma]\!]_\Gamma : [\![\Gamma]\!] \to [\![\sigma]\!]$ by:

- $[\![x^\sigma]\!]_{\Gamma;x:\sigma} = \pi_2$,
- $[\![y^\tau]\!]_{\Gamma;x:\sigma} = [\![y^\tau]\!]_\Gamma \circ \pi_1$,
- $[\![(st)^\tau]\!]_\Gamma = ev \circ \langle [\![s^{\sigma\to\tau}]\!]_\Gamma, [\![t^\sigma]\!]_\Gamma \rangle$,
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- $[\![(\lambda x.s)^{\sigma\to\tau}]\!]_\Gamma = \Lambda([\![s^\tau]\!]_{\Gamma;x:\sigma})$,
- $[\![(D(s)\cdot t)^{\sigma\to\tau}]\!]_\Gamma = \Lambda(\Lambda^-([\![s^{\sigma\to\tau}]\!]_\Gamma) \star [\![t^\sigma]\!]_\Gamma)$,
- $[\![0^\sigma]\!]_\Gamma = 0$,
- $[\![(s+S)^\sigma]\!]_\Gamma = [\![s^\sigma]\!]_\Gamma + [\![S^\sigma]\!]_\Gamma$.

Soundness

If **C** is a differential λ-category, then

$$Th_D(\mathbf{C}) = \{s = t \mid \Gamma \vdash_D s : \sigma \quad \Gamma \vdash_D t : \sigma \quad [\![s^\sigma]\!]_\Gamma = [\![t^\sigma]\!]_\Gamma\}$$

is a *differential λ-theory* (i.e., it contains $=_D$ and it is contextual).

We can interpret the Resource Calculus by translation:

$$[\![\Gamma \vdash_R M : \sigma]\!] = [\![(M^o)^\sigma]\!]_\Gamma$$

we get that $Th_R(\mathbf{C})$ is a *resource λ-theory.*

Theorem [BEM'10]

Differential λ-categories are sound models for:

- Simply Typed Differential λ-calculus
- Simply Typed Resource Calculus (by translation $(-)^o$)

Outline

Relational semantics - Example 1

MRel

- Objects: sets,
- Morphisms: $\mathbf{MRel}(A, B) = \mathcal{P}(\mathcal{M}_f(A) \times B)$ (relations between $\mathcal{M}_f(A)$ and B).

Given $f : A \to B$ we can define:

$$D(f) = \{((([a], m), b) \mid (m \uplus [a], b) \in f\} : A \times A \to B.$$

Theorem [BEM'10]

The category **MRel** is a differential λ-category.

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Theorem [BEM'10]

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Finiteness Spaces - Example 2

The category **MFin** of finiteness spaces and finitary morphisms.

Objects: finiteness spaces.

- $a, b \subset X$ are *orthogonal* $(a \perp b)$ if $a \cap b$ is finite.
- If $\mathcal{F} \subset \mathcal{P}(X)$ then $\mathcal{F}^\perp = \{b \in \mathcal{P}(X) \mid \forall a \in \mathcal{F}\ a \perp b\}$

Finiteness space

A finiteness space is a pair $\mathcal{X} = (X, F(\mathcal{X}))$, where

- X is a countable set,
- $F(\mathcal{X}) \subseteq \mathcal{P}(X)$ s.t. $F(\mathcal{X}) = F(\mathcal{X})^{\perp\perp}$.

Finiteness Spaces - Example 2

Morphisms: $f : \mathcal{X} \to \mathcal{Y}$ is a *finitary relation* from $!\mathcal{X} = (\mathcal{M}_f(X), F(!\mathcal{X}))$ to $\mathcal{Y}$, i.e., a relation $R \subseteq \mathcal{M}_f(X) \times Y$ s.t.:

- for all $a \in F(!\mathcal{X})$,
  $R(a) = \{\beta \in Y \mid \exists \alpha \in a \quad (\alpha, \beta) \in R\} \in F(\mathcal{Y})$, and
- for all $\beta \in Y$, $R^{\perp}(\beta) = \{\alpha \in X \mid (\alpha, \beta) \in R\} \in F(!\mathcal{X})^{\perp}$.

Theorem [BEM'10]

The category **MFin** of finiteness spaces is a differential λ-category.

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Finiteness Spaces - Actually Example $1\frac{1}{2}$

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Theorem [BEM'10]

The category **MFin** of finiteness spaces is a differential λ-category.

All categorical constructions are the same as in **MRel**, we just have to check that they are finitary/preserve finitary morphisms.

Outline

Conclusions

We have:

- Recalled the Resource Calculus + Type System R
- Recalled the Differential λ-Calculus + Type System D
- Shown Resource Calculus $\hookrightarrow$ Differential λ-Calculus
- Introduced the differential λ-categories,
  - Shown that they model both calculi (abstract definition),
  - **MRel** and **MFin** are instances.

Thank you for your attention!

Questions?